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On The Essential Set Of Function Algebras

By

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Let A be a function algebra on a compact Hausdorff space X , that is, A is a closed subalgebra of $C(X)$ which separates the points of X and contains the constants. In the following we shall present several results relating to the essential set of A , some of which are regarded as generalizations of the results published in several literatures [4], [5] and [8]. Complete proofs of these theorems and other details will be published elsewhere.

Throughout this paper M will indicate the maximal ideal space of A . The Šilov boundary of A will be denoted by ∂A . For a subset F in X , we shall denote by $A|F$ the restricted algebra of A to F . If $A|F$ is closed in $C(F)$, $A|F$ is regarded as a function algebra on F . A closed subset F in X is called an interpolation set for A if $A|F = C(F)$, and is called a ~~closed~~ restriction set if $A|F$ is closed in $C(F)$. Let G be an open set in X . G is called a w -interpolation set for A if any compact subset in G is an interpolation set for A .

Theorem 1. Let A be a function algebra on X and let $A \neq C(X)$. If G is any w -interpolation set for A , then $G \cap \partial_{A|E} = \emptyset$, where E is the essential set of A in X .

Corollary. Let A be a function algebra on X and suppose $E = \partial_{A|E}$. Then the set $X \sim E$ is the largest w -interpolation set for A .

The hypothesis of the corollary is necessary. Let X be the set consisting of the unit circle and the origin 0 in the unit disc and let A be the restriction of A_0 to X , where A_0 denotes the function algebra of all continuous functions on the unit circle which are analytic on the open unit disc. Then $E = X$. But we here see that $G = \{0\}$ is a w -interpolation set and $G \not\supseteq X \sim E = \emptyset$.

Bishop [3] and Glicksberg [6] have proved that A is characterized by the disjoint closed partitions of its maximal antisymmetric sets and Tomiyama [11] has shown that among these sets the set P of all maximal antisymmetric sets in X consisting of one point is free from the representing space X and plays a special rôle in determining the structure of the essential set E ; in fact

$$E = X \sim \text{int}(P),$$

where $\text{int}(P)$ is again free from the representing space X .

Now by the above mentioned result by Bishop and Glicksberg one easily sees that $\text{int}(P)$ is a w -interpolation set for A , and for each point $x \in \text{int}(P)$ one can find a closed neighborhood V of x such that $A|V = C(V)$. Mullins [8] has proved that the converse is true if $X = M_A$ is metrizable. Here we shall show, using Theorem 1, that the result is generally true so far as $X = M_A$.

Theorem 2. Let A be a function algebra on X with $X = M_A$ and let E be the essential set of A in X . Then, a point x belongs to the set $\text{int}(P)$ if and only if $A|V = C(V)$ for some closed neighborhood V of x .

Once we could succeed to generalize the result of Mullins [8; Theorem 1], the same idea which derives his Theorem 2 from

Theorem 1 would lead us to get the following.

Theorem 3. Let A be a function algebra on X with $X = M_A$ and let $\{F_j\}_{j=1}^{\infty}$ be a closed cover of X such that $A|_{F_j}$ is closed in $C(F_j)$ for each j . Then the closure of $\bigcup_{j=1}^{\infty} E_j$ is the essential set of A in X where E_j denotes the essential set of $A|_{F_j}$ in F_j .

Corollary. Let A be a function algebra on X . If X is covered by interpolation sets for A of countable number, then $A = C(X)$.

The above corollary has also been proved by Gamelin and Wilken [5], Chalice [4] and Mullins [8] (but [4] and [8] suppose more restricted conditions on A).

It is to be noticed that in theorem 2 and 3 one can not expect the same results for an arbitrary representing space X . The example cited after the corollary of Theorem 1 shows that the origin 0 satisfies the condition in Theorem 2 but belongs to the essential set. And, if we consider the cover $\{F_1, F_2\}$ of this representing space X as $F_1 = \{\text{unit circle}\}$ and $F_2 = \{0\}$, then $E_1 = \{\text{unit circle}\}$ and $E_2 = \emptyset$ and $E = E_1 \cup \{0\} \supsetneq E_1 = \overline{E_1}$. The reason which we could expect the above theorems is that in case $X = M_A$ the well known Šilov's theorem prevent us from facing the situation described in the above example. However, for some specialized function algebra one might expect this kind of generalization. Chalice [4], indeed, announced the results of this type. Including these results we shall present here more general results.

A function algebra A is said to be ε -regular on X for some (fixed positive number) ε if for each point x in X and each closed set F in X not containing x , there is a function f in A with $f(x) \neq 1$ and $|f| < \varepsilon$ on F . $|1 - f(x)| < \varepsilon$

(4)

Theorem 4. Let A be an ε -regular function algebra on X for $0 < \varepsilon \leq 1/2$. Then, a point x in X belongs to $\text{int}(P)$ if and only if $A|V = C(V)$ for some closed neighborhood V of x .

Corollary 1. If A is ε -normal function algebra on X for $0 < \varepsilon \leq 1/2$, the same conclusion holds.

Corollary 2. If A is approximately regular, still more approximately normal, function algebra on X the consequence in the theorem is also true.

Glicksberg [7] has proved the following theorem ; If any closed subset in X is a closed restriction set for A , then $A = C(X)$.

We moreover obtain the following

Theorem 5. Let A be a function algebra on X and let F_0 be a closed set in X . If $A|F_0$ is dense in $C(F_0)$ and if any compact subset F in $X \sim F_0$ is an interpolation set for A (or a closed restriction set for A), then $A = C(X)$.

Corollary. Let A be a function algebra on X . If F_0 is a closed set in X without perfect subsets (in particular, a closed countable set) and if any compact subset F in $X \sim F_0$ is an interpolation set for A (or a closed restriction set for A), then $A = C(X)$.

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